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COMMENT

Averages over percolation clusters on a Cayley tree

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Abstract. Averages over percolation clusters Γ of the form $\Sigma_\Gamma P(\Gamma)X(\Gamma)n(\Gamma)^k \equiv \chi_k(p)$ are considered, where $n(\Gamma)$ is the number of sites in the cluster, $X(\Gamma)$ is some property of the cluster and $P(\Gamma)$ is the probability per site of forming such a cluster at bond concentration p. Then

$$\chi_{k+1}(p) = \left(p(1-p)\frac{\partial \chi_k(p)}{\partial p} + (1+p)\chi_k(p)\right)(1-\sigma p)^{-1}$$

or

$$\chi_k(p) = \frac{(1-p)^2}{p}\int_0^p \frac{\chi_{k+1}(x)(1-\sigma x)}{x^3}\,\mathrm{d}x$$

where σ is the branching ratio of the Cayley tree.

This comment concerns averages over percolation clusters Γ of the form

$$\chi_k(p) \equiv \langle n^k(\Gamma)X(\Gamma)\rangle. \tag{1}$$

Here and below $X(\Gamma)$ is some quantity defined for a cluster Γ and $n(\Gamma)$ is the number of sites in the cluster Γ. For simplicity we assume that $X(\Gamma)$ depends only on the topology of the cluster Γ and not on its detailed shape. In that case, the average over clusters, $\langle\ \rangle$, can be expressed as a sum over topologically inequivalent clusters:

$$\langle X(\Gamma)\rangle = \sum_\Gamma P(\Gamma)W(\Gamma)X(\Gamma) \tag{2}$$

where $P(\Gamma) = p^{n_b(\Gamma)}(1-p)^{n_p(\Gamma)}$, where $n_b(\Gamma)$ and $n_p(\Gamma)$ are the number of bonds and the number of perimeter bonds respectively of the cluster Γ. In addition, $W(\Gamma)$, the weak embedding constant, is the number of clusters topologically equivalent to Γ which can be formed per site in the infinite lattice. When Γ is a single bond, $W(\Gamma) = z/2$, where z is the coordination number of the lattice.

There is emerging interest in averages of the type (1) for general values of k. For instance, Essam and Bhatti (1985) have shown that the resistive susceptibility χ_R (Harris and Fisch 1977) ($\chi_R \equiv \langle\Sigma_{i,j}R_{ij}\rangle$, where R_{ij} is the resistance between nodes i and j when each occupied bond has unit resistance) is related to the characteristic relaxation time for diffusion on percolation clusters:

$$\chi_R = 2\langle n(\Gamma)\sum_m{}' \lambda_m^{-1}(\Gamma)\rangle. \tag{3}$$

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Here $\lambda_m(\Gamma$ is the mth eigenvalue of the conductance or hopping matrix for the cluster Γ and the prime in (3) indicates omission of the term with $\lambda_m = 0$. Recently Harris *et al* (1987) have given expressions for averages of the form $\langle n^p(\Gamma)\Sigma'_m \lambda_m(\Gamma)^{-p}\rangle$ in terms of higher-order resistance correlations. In using such relations to calculate amplitude ratios on the Cayley tree (i.e. in order to reproduce mean-field theory), it is useful to know how these averages depend on the exponent to which $n(\Gamma)$ is raised. From the fact (Essam 1972) that $n(\Gamma)$ scales like $|p_c - p|^{-\Delta_p}$, where Δ_p is the gap exponent for percolation and p_c is the critical concentration, one can predict that, for $p \to p_c$,

$$\frac{\chi_{k+1}(p)}{\chi_k(p)} \sim A|p_c - p|^{-\Delta_p}.$$

Here we develop an exact relation for the Cayley tree between $\chi_{k+1}(p)$ and $\chi_k(p)$ from which A is determined.

From now on we consider a Cayley tree with branching ratio $\sigma = z - 1$. Adding a bond to a cluster increases the number of perimeter bonds by $(\sigma - 1)$. Thus in $\chi_k(p)$, p appears in the combination $p(1-p)^{\sigma-1}$ and $\chi_k(p)$ is of the form

$$\chi_k(p) = (1-p)^{\sigma+1} F[p(1-p)^{\sigma-1}]. \tag{4}$$

Thus we may write

$$F(x) = -\frac{1}{2\pi i}\oint \frac{\chi_k(z)}{(1-z)^{\sigma+1}}\left(\frac{(1-\sigma z)(1-z)^{\sigma-2}}{x - z(1-z)^{\sigma-1}}\right) dz. \tag{5}$$

To verify (5) note that the large round bracket is of the form $-f'(z)/f(z)$ where $f(z)$ has a simple pole where z satisfies $x = z(1-z)^{\sigma-1}$. We assume that x is small enough that only finite clusters exist. In that case, for $x = p(1-p)^{\sigma-1}$ only the pole at $z = p$ is relevant and the contour is chosen to surround this pole. Note that, since the contribution to $F(x)$ of order x^{n-1} comes exclusively from clusters of n sites, the quantity n can be generated by the operator $(\partial/\partial x)x$:

$$\chi_{k+1}(p) = (1-p)^{\sigma+1}\frac{\partial}{\partial x} xF(x)\bigg|_{x=p(1-p)^{\sigma-1}} \tag{6a}$$

$$= (1-p)^{\sigma+1}\frac{1}{2\pi i}\oint \frac{\chi_k(z)(1-\sigma z)z(1-z)^{\sigma-4}\,dz}{[p(1-p)^{\sigma-1} - z(1-z)^{\sigma-1}]^2}. \tag{6b}$$

The integrand in (6b) has a double pole at $z = p$ whose contribution yields the form for $\chi_{k+1}(p)$ quoted in the abstract. Ths relation can be regarded as a differential equation for $\chi_k(p)$ whose solution yields the equivalent integral relation quoted in the abstract for $\chi_k(p)$ in terms of $\chi_{k+1}(p)$.

As a simple application of the above results, take $X(\Gamma) = n(\Gamma)$ and $k = 0$ in (1). Then $\chi_0(p)$ is the probability that a site belongs to a finite cluster, so that $\chi_0(p) = 1$ for $p < p_c = \sigma^{-1}$. Use of the relation in the abstract for $\chi_{k+1}(p)$ yields the well known result for the percolation susceptibility, $\chi_1(p) \equiv \langle n^2(\Gamma)\rangle$:

$$\chi_1(p) = (1+p)/(1-\sigma p). \tag{7}$$

Also, if $\chi_k(p)$ is singular, i.e. if $\chi_k(p) \sim A|p_c - p|^{-x}$ with $x > 0$, then one has

$$\chi_{k+1}(p) \sim A\frac{xp(1-p)}{(1-\sigma p)}|p - p_c|^{-x-1} \tag{8a}$$

$$\sim Ax\frac{(\sigma-1)}{\sigma^3}|p - p_c|^{-x-2} \tag{8b}$$

consistent with the mean-field gap exponent, $\Delta_p = 2$. From this result one can obtain universal amplitude ratios (Aharony 1980, Adler *et al* 1986), such as

$$\frac{\chi_{k+1}(p)\kappa_{k-1}(p)}{\chi_k(p)^2} \sim \frac{x}{x-2}. \tag{9}$$

This result applies throughout the regime in which mean-field theory is valid, i.e. for $d > 6$ (Toulouse 1974, Harris *et al* 1975).

Using (5) one can generate expressions for $x(N) \equiv \langle x(\Gamma)\delta_{n(\Gamma),N}\rangle$, the value of X when the sum in (2) is restricted to clusters of exactly N sites. To get $X(N)$ from $\chi_0(p)$, one needs to isolate the contribution to $F(x)$ in (5) of order x^{N-1}. Then use of (4) yields

$$X(N) = p^{N-1}(1-p)^{N\sigma - N + 2} f(N) \tag{10}$$

with

$$f(N) = \frac{1}{2\pi \mathrm{i}} \oint \frac{\chi_0(z)(1-\sigma z)}{(1-z)^{N\sigma - N + 3} z^N} \, \mathrm{d}z \tag{11}$$

where the contour surrounds $z = 0$. For instance, if we take $X(\Gamma)$ to be $\Sigma_{i,j} R_{ij}$ so that $\chi_0(p)$ is the resistive susceptibility, $\chi_0(p) = (\sigma+1)p(1-\sigma p)^{-2}$, then one finds

$$f(N) = (\sigma+1) \sum_{k=0}^{N-2} \frac{(N\sigma - k)!}{(N-2-k)!(N\sigma - N + 2)!} \sigma^k. \tag{12}$$

For small values of N one can check explicitly that $f(N)$ is indeed the value of the resistive susceptibility summed over diagrams with N sites. More generally, suppose that $\chi_0(p) = A(1-\sigma p)^{-x}$. Then for large N, (11) gives the result

$$f(N) = \frac{A\lambda^{N-1}}{2} \left(\frac{\sigma}{\sigma - 1}\right)^{\sigma+2} \left(\frac{N\sigma}{2(\sigma-1)}\right)^{(x/2)-1} \frac{1}{[(x/2)-1]!} \tag{13}$$

where $\lambda = \sigma^\sigma/(\sigma-1)^{\sigma-1}$. If we divide $f(N)$ by the total number $C(N)$ of a cluster of size N (Fisher and Essam 1961), $C(N) = (\sigma+1)(N\sigma)!/[N!(N\sigma - N + 2)!]$, we obtain the average of X over clusters of size N as

$$\langle X \rangle_N = \frac{\sqrt{\pi} N^2 A}{[(x/2)-1]!\,(1+\sigma^{-1})} \left(\frac{N\sigma}{2(\sigma-1)}\right)^{(x-1)/2}. \tag{14}$$

For the percolation susceptibility (7) this gives, with $x = 1$, $A = (1+\sigma^{-1})$, $\langle N^2 \rangle_N = N^2$, as expected.

In summary, use of the relations given in the abstract allows one to evaluate exactly certain universal amplitude ratios within mean-field theory. Use of (14) allows one to calculate averages on the Cayley tree restricted to clusters of N sites.

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